PET ENGINEERING COLLEGE

An ISO 9001:2015 Certified Institution

Accredited by NAAC, Approved by AICTE, Recognized by Government of Tamil Nadu and Affiliated to Anna University

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

UNIT – II

TIME RESPONSE ANALYSIS

CLASS   : **S3 ECE**

SUBJECT CODE   : **EC3351**

SUBJECT NAME   : **CONTROL SYSTEM**
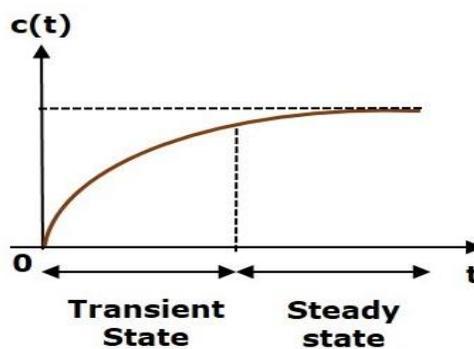
REGULATION   : **2021**

UNIT 2
TIME RESPONSE ANALYSIS

Time Response:

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure.

Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t)=c_{tr}(t)+c_{ss}(t)c(t)$$

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response
-

Response of the First Order System

Time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1sT1sT is connected with a unity negative feedback.
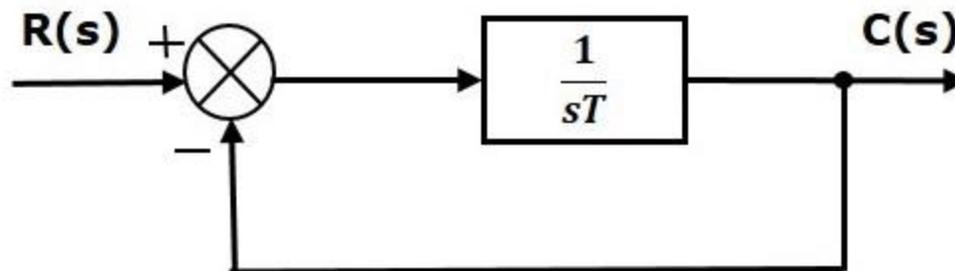
We know that the transfer function of the closed loop control system has unity negative feedback as,

$$C(s)/R(s)=G(s)/1+G(s)$$

Substitute, $G(s)=1sT$ in the above equation.
$$C(s)/R(s)=1sT1+1sT=1sT+1$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.We can re-write the above equation as

$$C(s)=(1sT+1)R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),

- **R(s)** is the Laplace transform of the input signal r(t), and

- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal r(t)r(t).
- Consider the equation, C(s)=(1sT+1)R(s)C(s)=(1sT+1)R(s)
- Substitute R(s)R(s) value in the above equation.
- Do partial fractions of C(s)C(s) if required.
- Apply inverse Laplace transform to C(s)C(s).

Step Response of First Order System

- Consider the **unit step signal** as an input to first order system.

$$\text{So, } r(t) = u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s(sT+1)} = \frac{A(sT+1) + Bs}{s(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Substitute, A = 1 and B = −T in partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{T}{sT + 1} = \frac{1}{s} - \frac{T}{T\left(s + \frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

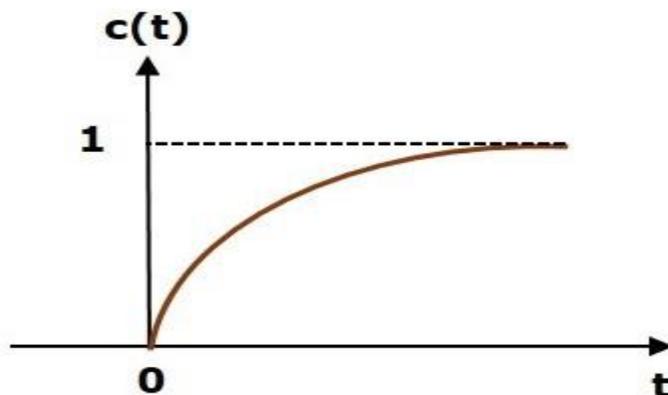The **unit step response**, c(t) has both the transient and the steady state terms.

The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit step response is -

$$c_{ss}(t) = u(t)$$

The following figure shows the unit step response.

- The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Response of Second Order System

Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\frac{\omega_n^2}{s(s+2\delta\omega_n)}$ is connected with a unity negative feedback.
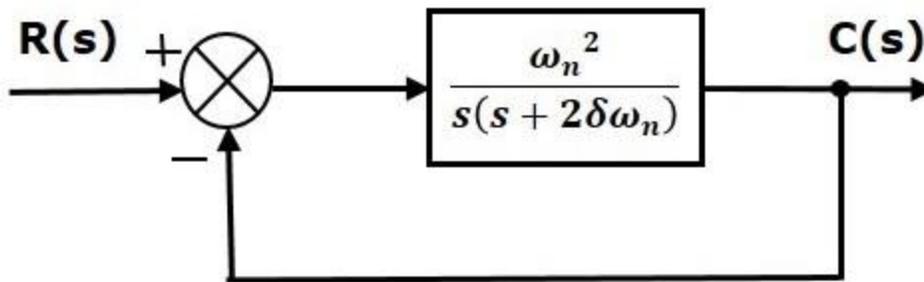
We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is –

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are –

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

- The two roots are imaginary when δ = 0.
- The two roots are real and equal when δ = 1.
- The two roots are real but not equal when δ > 1.
- The two roots are complex conjugate when 0 < δ < 1.

We can write C(s)C(s) equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$$

.

Where,

- **C(s)** is the Laplace transform of the output signal, c(t)

- **R(s)** is the Laplace transform of the input signal, r(t)

- **ωn** is the natural frequency

- **δ** is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain.

- Take Laplace transform of the input signal, r(t)
- Consider the equation,

$$C(s) = \left(\frac{w_n^2}{s^2 + 2\delta w_n s + w_n^2} \right) R(s)$$

- Substitute R(s)R(s) value in the above equation.
- Do partial fractions of C(s)C(s) if required.
- Apply inverse Laplace transform to C(s)C(s).

Step Response of Second Order System

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$$R(s)=1sR(s)=1s$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2\delta w_n s + w_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + w_n^2}$$

$$\Rightarrow C(s) = \left(\frac{w_n^2}{s^2 + w_n^2} \right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{w_n^2}{s^2 + w_n^2} \right) \left(\frac{1}{s} \right) = \frac{w_n^2}{s(s^2 + w_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(w_n t)) \, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{s^2 + 2w_n s + w_n^2}$$

$$\Rightarrow C(s) = \left(\frac{w_n^2}{(s + w_n)^2} \right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{w_n^2}{(s + w_n)^2} \right) \left(\frac{1}{s} \right) = \frac{w_n^2}{s(s + w_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{w_n^2}{s(s + w_n)^2} = \frac{A}{s} + \frac{B}{s + w_n} + \frac{C}{(s + w_n)^2}$$

After simplifying, you will get the values of A, B and C as $1, -1 \, and \, -w_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s + w_n} - \frac{w_n}{(s + w_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-w_n t} - w_n t e^{-w_n t}) u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows –

$$s^2 + 2\delta w_n s + w_n^2 = \{s^2 + 2(s)(\delta w_n) + (\delta w_n)^2\} + w_n^2 - (\delta w_n)^2$$

$$= (s + \delta w_n)^2 + w_n^2(1 - \delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{w_n^2}{(s + \delta w_n)^2 + w_n^2(1 - \delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{w_n^2}{(s + \delta w_n)^2 + w_n^2(1 - \delta^2)} \right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{w_n^2}{(s + \delta w_n)^2 + w_n^2(1 - \delta^2)} \right) \left(\frac{1}{s} \right) = \frac{w_n^2}{s \left((s + \delta w_n)^2 + w_n^2(1 - \delta^2) \right)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{w_n^2}{s \left((s + \delta w_n)^2 + w_n^2(1 - \delta^2) \right)} = \frac{A}{s} + \frac{Bs + C}{(s + \delta w_n)^2 + w_n^2(1 - \delta^2)}$$

After simplifying, you will get the values of A, B and C as $1, -1 \, and \, - 2\delta w_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} - \frac{\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2+(\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+\omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2+\omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin(\omega_d t)\right)u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right)u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then '$\delta$' will be $\cos(\theta)$. Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right)u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Time Domain Specifications

The time domain specifications of the second order system. The step response of the second order system for the underdamped case is shown in the following figure.
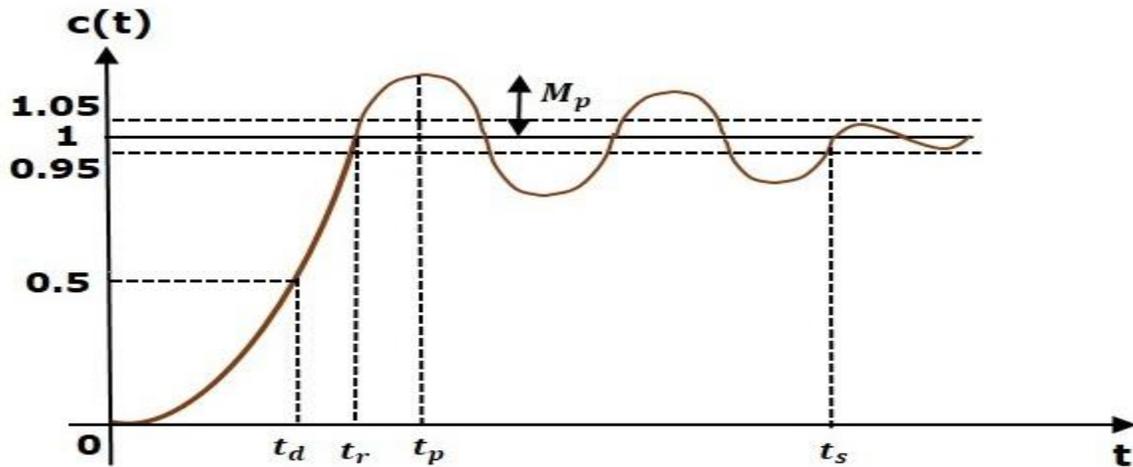
All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by tdtd.

Consider the step response of the second order system for t ≥ 0, when 'δ' lies between zero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by **t_r**.

At $t = t_1 = 0$, $c(t) = 0$.

We know that the final value of the step response is one.

Therefore, at t=t2, the value of step response is one. Substitute, these values in the following equation.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = \pi$$

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r = t_2 - t_1$$

$$\therefore t_r = \frac{\pi - \theta}{\omega_d}$$

From above equation, we can conclude that the rise time t_r and the damped frequency ω_d are inversely proportional to each other.

3.   ***Peak time tp:*** It is the time required for the response to reach the maximum or Peak value of the response.

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\delta^2}}$$

4.   **Peak overshoot M** : It is defined as the difference between the peak value of the response and the steady state value. It is usually expressed in percent of the steady state value. If the time for the peak is *tp'* percent peak overshoot is given by,

$$M_p = 100 \; e^{\frac{-\pi\delta}{\sqrt{1-\delta^2}}} \; \%$$

5.Settling time ts : It is the time required for the response to reach and remain within a specified tolerance limits (usually ± 2% or ± 5%) around the steady state value.

$$t_s \simeq \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}$$

6.   **Steady state error ess** : It is the error between the desired output and the actual output as $t \sim 00$ or under steady state conditions. The desired output is given by the reference input *r (t)* and c(t).

$$e_{ss} = \underset{t \to \infty}{Lt} \; [r(t) - c(t)]$$

P, PI and PID Controllers

- The controller (an analogue/digital circuit, and software), is trying to keep the controlled variable such as temperature, liquid level, motor velocity, robot joint angle, at a certain value called the **set point (SP)**.
- A feedback control system does this by looking at the **error (E)** signal, which is the difference between where the controlled variable (called the **process variable (PV)**) is, and where it should be.
- Based upon the error signal, the controller decides the magnitude and the direction of the signal to the actuator.

The proportional (P), the integral (I), and the derivative (D), are all basic controllers.

Types of controllers: P, I, D, PI, PD, and PID controllers

- **Proportional Control**

With proportional control, the actuator *applies a corrective force that is proportional to the amount of error*:

$$Output_p = K_p \times E$$

$Output_p$ = system output due to proportional control

K_p = proportional constant for the system called **gain**

E = error, the difference between where the controlled variable should be and where it is. $E = SP - PV.$